Transition from stable to unstable growth by an inertial force

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We introduce a simple growth model where the growth of the interface is affected by an inertial force and a white noise. The magnitude of the inertial force is controlled by a constant p between 0 and 1. An inertial force increases continuously from 0, as p does from 0 to 1. In our model, the interface starts growing from a flat state. When $p<p_c$, the interface width in our model increases continuously from 0 as time elapses, but it saturates to a constant value in the long time limit. The saturated values of the interface width are the same for different values of p if $p<p_c$. When $p>p_c$, however, the interface width increases continuously without saturation as time elapses. We explain via simple calculation how this interesting phenomenon occurs in our model. We find $p_c=0.5$ from the calculation. This critical value is in excellent agreement with the critical value $p_c=0.50(1)$ found from the simulations of our model.

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The study of a growing interface in disordered media has been a popular research topic for the last decade because it relates to various physical systems such as interface growth in porous media [1,2], charge density waves under external fields [3–5], fluid imbibition in paper [6], driven flux motion in type-II superconductors [7,8], etc. It is well known that an interface driven by an external driving force F through random media with weak disorder shows a pinning-depinning (PD) transition at F_c from a moving phase with nonzero growth velocity V to pinned phase with $V=0$. This PD transition is a continuous phase transition. But recent experiments [9–11] reported that a driven interface in a system with strong disorder shows a history-dependent depinning transition, i.e., a first-order depinning transition. The interface driven through strong disorder exhibits a spatially inhomogeneous plastic response without long-wavelength elastic restoring force, which happens in a system with weak disorder. This inhomogeneous plastic response invokes the first-order transition of a growing interface.

Marchetti, Middleton, and Prellberg [12] succeeded in showing a first-order depinning transition from a simple coarse-grained model, which mimics the interface growth in a system with strong disorder. Schwarz and Fisher [13] also studied the critical behavior of the growing interface showing a first-order depinning transition from a mean-field model. Above two studies are all about a driven first-order depinning transition in the presence of quenched disorder. In the two studies, *an inertial force* was introduced to describe a spatially inhomogeneous plastic response in the growing interface. Both models were defined, in general, for finite-range interaction. The inertial force enters as a coupling to the local velocity of the system. However, the inertial force enters as a global coupling to the mean velocity of the system in the mean-field theory, which is formally valid in the limit of infinite-range interaction. Recently these two works have been partially extended to finite dimensions [14].

Recently Park *et al.* [15] showed from the simulations of a simple growth model that a first-order depinning transition occurs when the local inertial force $pL\overline{v}$ is involved in the interface growth in disordered media. Here $\overline{v}$ is an average velocity in a local region of a growing interface and L is the system size. The growth model shows two phase transitions, a second-order and a first-order phase transitions, depending on the value of p. When $p<p_c$, the model shows a second-order PD transition at $F_c(p)$. But when $p>p_c$, the model shows a first-order PD transition at $F_c(p)$.

From recent studies [12,13,15], one knows that an inertial force changes dramatically the dynamical behavior of a growing interface in disordered media. Until now, however, all studies about the effect of an inertial force have been done for growing interfaces in disordered media. Therefore, it is very interesting to study the effect of an inertial force for the growing interface in a homogeneous medium. To this end, we study a simple growth model mimicking the interface growth in a homogeneous medium, where the growth of the interface is affected by an inertial force and a white noise.

Our model is defined on a $(1+1)$-dimensional lattice with periodic boundary conditions (see Fig. 1). In the model, a particle is deposited at a chosen site i on the substrate at each time step. The deposited particle is allowed to diffuse to one of its two nearest neighbor sites $(i-1,\ i+1)$ to find the position with the lowest height. We calculate $n_j(t)$ with $j=1,\ldots,L$ for all sites in the system after the interface growth occurs. $n_j(t)$ is defined as follows: $n_j(t)$ is zero at $t=0$, i.e., $n_j(0)=0$. If a particle is deposited at a site i and diffuses to a site $i+1$ at time t, $n_i(t)$ and $n_{i+1}(t)$ become $n_i(t-1)+1$ and $n_{i+1}(t-1)+1$, respectively. Then, at every site except i and $i+1$ in the system, $n_j(t)$ becomes $n_j(t-1)-1$. If a particle is deposited at a site i and there is no diffusion of the particle, then $n_i(t)=n_i(t-1)+1$ and $n_j(t)$ becomes $n_j(t-1)-1$ at every site except i in the system.

The inertial force at a site j is defined as $p\Theta(n_j(t))$, where p is a constant between 0 and 1. The unit-step function $\Theta(n_j(t))$ can have 0 or 1 according to the value of $n_j(t)$, i.e., $\Theta(n_j(t))=1$ for $n_j(t)\geqslant 0$ and $\Theta(n_j(t))=0$ for $n_j(t)<0$. Each time we assign a new random number between 0 and 1 at every site on the interface and then calculate

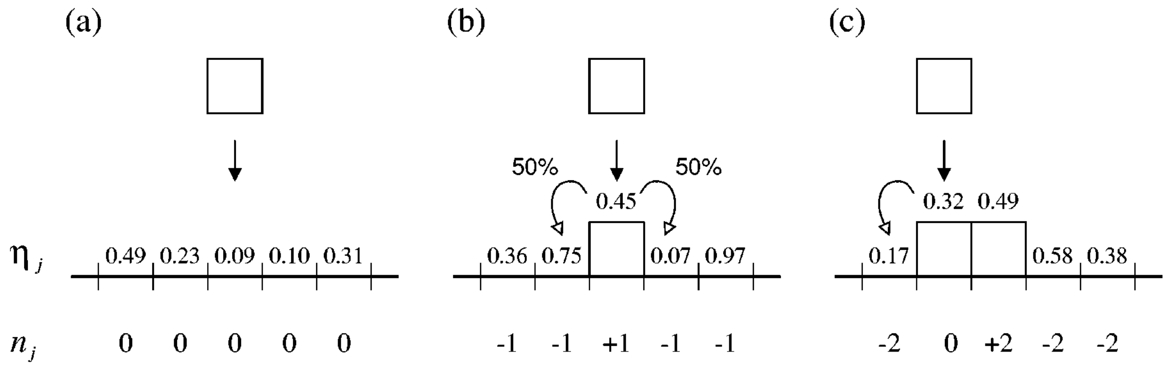

FIG. 1. Schematic representations of the growth rule of our model. (a) The flat interface at $t=0$. At $t=1$, the deposition of a particle occurs at the site having the lowest minimum random number because of $n_j(0)=0$ at every site in the system. (b) The interface at $t=1$. At $t=2$, the deposition of a particle occurs at a site having the lowest minimum number among s_j's. The deposited particle diffuses to the left or right with 50% probability to find the position with the lowest height. (c) The interface at $t=2$. At $t=3$, the deposition of a particle occurs at the site having the lowest minimum number among s_j's. The deposited particle diffuses to the left.

$$s_j=\eta_j-p\Theta(n_j(t)) \tag{1}$$

at every site j in the system. Here η_j means a random number assigned at a site j. Each time a particle is deposited at the site having the lowest minimum number among s_j's in the system. As p increases from 0 to 1, the deposition of a particle occurs more often at the site where the deposition or diffusion of a particle occurred just before.

When $p=0$, the growth rule of our model is the same as the Family model [16]. It is known that the dynamics of the Family model can be well described by the Edwards-Wilkinson (EW) equation [17]

$$\frac{\partial h(x,t)}{\partial t}=\nu\nabla^2 h(x,t)+\eta(x,t), \tag{2}$$

where $h(x,t)$ denotes the height of the interface at position x and time t. $\nu\nabla^2 h(x,t)$ describes the smoothening effect of the interface tension. η is a white noise with $\langle\eta(x,t)\rangle=0$ and $\langle\eta(x,t)\eta(x',t')\rangle=2D\delta^{d'}(x-x')\delta(t-t')$. Here d' denotes the substrate dimension.

The dynamics of the growing interface formed by the EW equation show a nontrivial scaling behavior in the interface width,

$$W(L,t)=\left\langle\frac{1}{L^{d'}}\sum_x\,[h(x,t)-\overline{h}(t)]^2\right\rangle^{1/2}, \tag{3}$$

where $\overline{h}$ denotes the mean height. The interface width scales as

$$W(L,t)\sim\begin{cases}t^{\beta} & \text{if}\quad t\ll L^z\\ L^{\zeta} & \text{if}\quad t\gg L^z.\end{cases} \tag{4}$$

The exponents ζ, β, and z are called the roughness, the growth, and the dynamic exponent, respectively. These exponents are related by $z\beta=\zeta$.

The scaling exponents of the EW equation can be obtained easily by solving the equation directly [18]. The obtained scaling exponents are

$$\zeta=\frac{2-d'}{2},\quad \beta=\frac{2-d'}{4},\quad z=2. \tag{5}$$

We carried out computer simulations of our model for the system size $L=512$ by increasing p from 0 to 1. The numerical data were averaged over more than 200 configurations. We found that the interface width in our model saturates to a constant value for $p<p_c[=0.50(1)]$ and $t\gg L^z$, but it increases continuously without saturation for $p>p_c$ (see Fig. 2).

In order to obtain the growth exponent for our model, we measure the time-dependent behavior of the interface width $W(L,t)$ starting from an initially flat interface. We plot $W(L,t)$ versus time t in double logarithmic scales in Fig. 2. When $p=0$, the growth exponent is estimated as $\beta=0.25(1)$. This value is in excellent agreement with that expected from the EW equation. For $0<p<p_c$, we cannot

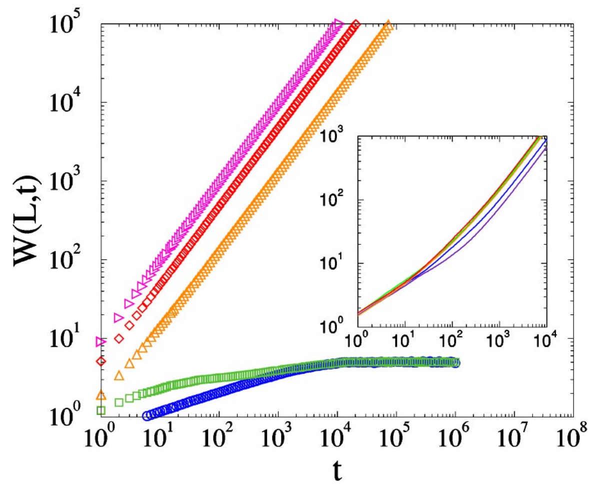

FIG. 2. The plot of $W(L,t)$ vs time t in double logarithmic scales at $p=0.70$, 0.60, 0.51, 0.49, and 0 from top to bottom for the system size $L=512$. Inset: The plot of $W(L,t)$ vs time t just above p_c in double logarithmic scales at $p=0.5005$, 0.5004, 0.5003, 0.5002, and 0.5001 from top to bottom for the system size $L=1024$.

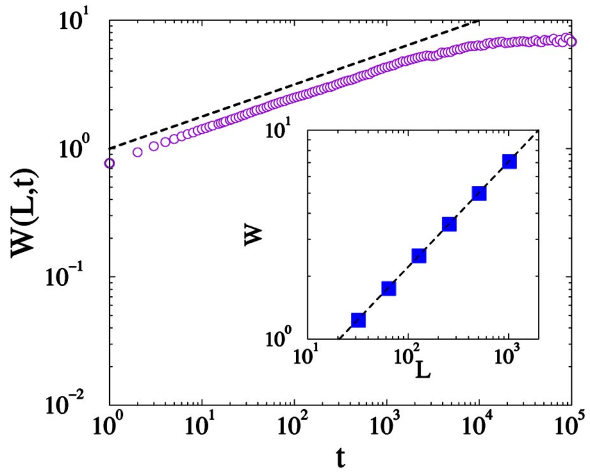

FIG. 3. The plot of $W(L,t)$ vs time t in double logarithmic scales for $p=0.49$ and $L=512$. $W(L,t)$ was measured starting from the saturated interface width. The guideline is for $\beta_s=0.25$. Inset: The plot of the saturated value of $W(L,t)$ vs the system size L in double logarithmic scales at $p=0.49$. The system sizes are $L=32$, 64, 128, 256, 512, and 1024. The guideline is for $\zeta=0.5$.

measure the growth exponent because the interface width does not show a straight line in the log-log plot for $t \ll L^z$ (see Fig. 2). We also consider another growth exponent β_s by measuring the interface width from the saturated interface instead of the flat interface. The growth exponent is measured as $\beta_s=0.25(1)$ for $p<p_c$ (see Fig. 3). This value is in good agreement with that obtained by solving the EW equation. We found that the saturated values of the interface width are the same for different p if $p<p_c$. However, the interface width increases with $\beta=1$ without saturation for $p>p_c$. As shown in the inset of Fig. 2, the interface widths start to increase continuously just above p_c. Similar dynamical behavior was observed from the study of an inertial sandpile model where the distribution of avalanche sizes was studied instead of the interface roughening [19].

In order to obtain the roughness exponent for $p<p_c$, we plot the saturated value of $W(L,t)$ versus the system size L in double logarithmic scales. The obtained roughness exponent is $\zeta=0.50(1)$ (see the inset of Fig. 3), which is in excellent agreement with the value expected from the EW equation.

We found from the simulations of our model that $n_j(t)$ at every site in the system has a negative value in the long time limit for $p<p_c$. If $n_j(t)<0$ at every site in the system, the site where the deposition of a particle occurs is determined only by a white noise without any affection of the inertial force. In that case, the probability that $n_j(t)$ becomes $n_j(t-1)+1$ is $1/L$ at every site in the system. But the probability that $n_j(t)$ becomes $n_j(t-1)-1$ is $(L-1)/L$ at every site. In the limit $L\to\infty$, therefore, the probability that $n_j(t)$ becomes $n_j(t-1)-1$ is one at every site in the system. Therefore, it is impossible that $n_j(t)$ becomes 0 or positive for $p<p_c$, since it has a large negative value in the limit $L\to\infty$. However, we found that, when $p>p_c$, $n_j(t)$ has a very large positive value at a certain site i and has a very large negative value at every other site except i in the system in the long time limit. In other words, the deposition of a particle occurs repeatedly only at a certain site i after some initial growth if $p>p_c$. If the deposition occurs successively only at a site i, then the value of $n_j(t)$ becomes a very large positive one at site i, but becomes a very large negative one at all other sites except i in the system in the long time limit. In that case, the growth of the interface occurs only at three sites $(i-1,i,i+1)$. The interface growth at the two sites $i-1$, $i+1$ occurs because of the diffusion of a deposited particle at site i.

We can explain the above simulation results very well via simple calculation. Let us assume that $n_j(t-1)$ has a large positive value at a site i, but has a large negative value at all other sites except i in the system at time $t-1$. The value of $n_i(t)$ at time t can be obtained as follows:

$$n_i(t)=n_i(t-1)+G(p,L), \tag{6}$$

where $G(p,L)$ is a function defined as $G(p,L)=1$ for $p>\eta_i$ and $G(p,L)=-1$ for $p<\eta_i$ in the limit $L\to\infty$. If $p>\eta_i$, then $s_i[=\eta_i-p\Theta(n_i(t-1))]<0$. s_j has a negative value only at site i and a positive value at all other sites except i. Therefore, the deposition of a particle at time t occurs at site i. $n_i(t)$ becomes $n_i(t-1)+1$ and $n_j(t)$ becomes $n_j(t-1)-1$ at every site except $i-1$, i, and $i+1$. $n_k(t)$ with $k=i-1,i+1$ becomes $n_k(t-1)-1$ with probability 2/3 and $n_k(t-1)+1$ with probability 1/3 because of the diffusion of a deposited particle. But if $p<\eta_i$, s_j at every site including i is positive. Then the probability for a particle to be deposited at site i becomes smaller and smaller as the system size L increases. The probability becomes almost 0 when $L\to\infty$. That is the reason why we define $G(p,L)=-1$ if $p<\eta_i$ and $L\to\infty$. Now, we can calculate the expectation value of $n_i(t)$, $\overline{n_i}(t)$, as follows

$$\begin{aligned}
\overline{n_i}(1+t_0)&=\overline{n_i}(t_0)+\overline{G}(p,L),\\
\overline{n_i}(2+t_0)&=\overline{n_i}(1+t_0)+\overline{G}(p,L),\\
&\vdots\\
\overline{n_i}(t+t_0)&=\overline{n_i}(t-1+t_0)+\overline{G}(p,L),
\end{aligned} \tag{7}$$

where $\overline{G}(p,L)$ means the expectation value of $G(p,L)$.

From Eq. (7), we get

$$\overline{n_i}(t+t_0)=\overline{n_i}(t_0)+(t-1)\overline{G}(p,L), \tag{8}$$

where we assumed that $n_j(t_0)$ has a large positive value at site i and has a large negative value at all other sites except i in the system. One can easily find that the expectation value of $G(p,\infty)$ becomes 1 with probability p, and -1 with $1-p$. Therefore, the expectation value of $n_i(t+t_0)$ can be written as

$$\begin{aligned}
\overline{n_i}(t+t_0)&=\overline{n_i}(t_0)+(t-1)[p-(1-p)]\\
&=n_i(t_0)+(t-1)(2p-1).
\end{aligned} \tag{9}$$

When $p<0.5$, $\overline{n_i}(t)$ always has a negative value for large t although $\overline{n_i}(t_0)$ has a positive value. This result says that $\overline{n_j}(t)$ at every site in the system becomes negative for large t regardless of the initial value of $n_j(t_0)$. In that case, the deposition of a particle occurs at a randomly selected site on the interface. The growth rule of our model is the same as that of the Family model after some initial growth if $p<p_c$. Hence, the dynamics of the growing interface can be described well by the EW equation. The interface width saturates to a constant value after some initial increase. When $p>0.5$, however, $\overline{n_i}(t)$ always has a positive value and increases as time goes on. $\overline{n_j}(t)$ at every site except i in the system has a negative value and decreases continuously as time goes on. In that case, the deposition of a particle occurs only at a site i. The growth of the interface occurs only at three sites $(i-1,i,i+1)$. Then the interface width increases continuously without saturation. From this calculation, we know that a phase transition from stable to unstable growth occurs at $p_c=0.5$. Here stable growth means the growth where the interface width saturates to a constant value after some initial increase. On the other hand, unstable growth means the growth where the interface width increases continuously without saturation. The critical point obtained from our simple calculation is in excellent agreement with that found from the computer simulations of our model.

We can also easily prove $\beta=1$ for $p>p_c$. The interface grows at only three sites $(i-1,i,i+1)$ in the long time limit if $p>p_c$. Then $h_j(t)\sim t/3$ at three sites $j=i-1,i,i+1$, $h_j(t)\sim$const at other sites, and $\overline{h}(t)\sim t/L$ in the limit $L\to\infty$ and $t\to\infty$. Therefore, the interface width in 1+1 dimensions can be calculated as follows. $W^2(L,t)=(1/L)\Sigma_j[h_j(t)-\overline{h}(t)]^2\sim(1/L)[3(t/3-t/L)^2+\Sigma_k(\text{const}-t/L)^2]\sim t^2$, where $k\neq i-1,i,i+1$.

In conclusion, we have introduced a simple growth model where the interface growth is affected by an inertial force $p\Theta(n_j)$ and a white noise. When $p<p_c$, the interface width increases continuously from 0 as time goes on, but saturates to a constant value when $t\gg L^z$. Then all saturated values of the interface width for different p are the same if $p<p_c$. But when $p>p_c$, the interface width increases continuously without saturation as time goes on. We explained such nontrivial dynamical behavior of our model via simple calculation. From the calculation, we found that there exists a critical point $p_c=0.5$ in our model. For $p<p_c$, the dynamics of our model is described by the EW type growth. Then the dynamics of the growing interface is not affected by the inertial force. But for $p>p_c$, the dynamics of the growing interface is affected only by the inertial force without any affection of the white noise. Therefore, the interface width grows continuously without saturation.

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